Introduction to Mathematical Quantum Theory

Solution to the Exercises

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Exercise 1

Let $\mathcal{H}$ be a Hilbert space. Let V any closed subspace of $\mathcal{H}$; recall the definition of $V^\perp$ as

$$V^\perp := \{ f \in \mathcal{H} \mid \langle g, f\rangle = 0 \ \forall g \in V \} . \tag{1}$$

We saw in class that the Hilbert space $\mathcal{H}$ can be decomposed as $\mathcal{H} = V \oplus V^\perp$, meaning that $V \cap V^\perp = \{0\}$ and that for any non-zero $f \in \mathcal{H}$ there exists a unique element $f_V \in V$ such that $f - f_V \in V^\perp$. Define $P_V f := f_V$; from the uniqueness of f_V this is a well defined linear mapping.

- **a** Prove that $P_V^2 = P_V = P_V^*$.
- **b** Use **a** to prove that P_V is bounded and if $V \neq \{0\}$ then $\|P_V\| = 1$.
- **c** Prove that if V_1 and V_2 are two closed subspaces of $\mathcal{H}$ then[1]

$$V_1 \perp V_2 \qquad \Longleftrightarrow \qquad P_{V_1} P_{V_2} = 0. \tag{2}$$

Proof. We first prove that $P_V^2 = P_V$. To prove this is enough to notice that if $f \in V$ then $P_V f = f$. Indeed, let $g := f - P_V f$. Then by definition $g \in V^\perp$. On the other hand, both f and $P_V f$ are in V, therefore $g \in V \cap V^\perp = \{0\}$ and this implies $P_V f = f$. Now from the fact that $P_V f \in V$ for any $f \in \mathcal{H}$ we conclude that $P_V^2 f = P_V f$.

To prove that $P_V^* = P_V$, first notice that we have the trivial identity $\mathrm{id} = P_V + (\mathrm{id} - P_V)$. Moreover, by definition of P_V and from the decomposition $\mathcal{H} = V \oplus V^\perp$ we get that $(\mathrm{id} - P_V)(\mathcal{H}) \subseteq V^\perp$. Consider now $f,\ g \in \mathcal{H}$. We then have

$$\begin{aligned}
\langle g, P_V^* f\rangle &= \langle P_V g, f\rangle \\
&= \langle P_V g, P_V f\rangle + \langle P_V g, (\mathrm{id} - P_V) f\rangle \\
&= \langle P_V g, P_V f\rangle \\
&= \langle g, P_V f\rangle - \langle (\mathrm{id} - P_V) g, P_V f\rangle \\
&= \langle g, P_V f\rangle.
\end{aligned}$$

From the fact that this is true for every $f,\ g \in \mathcal{H}$ we get that $P_V^* = P_V$.

To prove **b** for any $f \in \mathcal{H}$ we get that

$$\begin{aligned}
\|f\|^2 &= \langle f, f\rangle \\
&= \langle P_V f, f\rangle + \langle (\mathrm{id} - P_V) f, f\rangle \\
&= \langle P_V f, P_V f\rangle + \langle (\mathrm{id} - P_V) f, (\mathrm{id} - P_V) f\rangle \\
&= \|P_V f\|^2 + \|(\mathrm{id} - P_V) f\|^2 .
\end{aligned}$$

[1] We denote with $\perp$ the condition of two subspaces of an Hilbert space $\mathcal{H}$ of being orthogonal, i.e., V_1 is orthogonal to V_2, or $V_1 \perp V_2$ if and only if for any $(f, g) \in V_1 \times V_2$ we have $\langle f, g\rangle = 0$.

From this we can deduce that P_V is bounded and that $\|P_V\| \leqslant 1$. If V is non empty, let $f \in V$, $\|f\| = 1$; then $\|P_V f\| = \|f\| = 1$ and this implies that $\|P_V\| = 1$.

To prove **c** first suppose $V_1 \perp V_2$ and $f \in V_2$. By definition of P_{V_1} we have that $f - P_{V_1} f \in V_1^{\perp}$; then we get that

$$P_{V_1} f = f - (f - P_{V_1} f) \in V_1 \cap V_1^{\perp} \Rightarrow P_{V_1} f = 0.$$

Consider now $f \in \mathcal{H}$; given that $P_{V_2} f \in V_2$ we can deduce that $P_{V_1} P_{V_2} = 0$.

Suppose now that $P_{V_1} P_{V_2} = 0$. Consider now $f \in V_1$, $g \in V_2$. Then we have

$$\langle f, g\rangle = \langle P_{V_1} f, P_{V_2} g\rangle = \langle f, P_{V_1} P_{V_2} g\rangle = 0.$$

Given that f and g were generic this implies that $V_1 \perp V_2$.

□

Exercise 2

Let $\phi(t)$ and $\psi(t)$ differentiable functions on the Hilbert space $\mathcal{H}$, meaning that the limit

$$\frac{d\phi}{dt}(t) := \lim_{h \to 0} \frac{\phi(t+h) - \phi(t)}{h} \tag{3}$$

exists in the norm topology of $\mathcal{H}$ for each $t \in \mathbb{R}$, and similarly for $\psi(t)$.

Prove that

$$\frac{d}{dt}\langle \phi(t), \psi(t)\rangle = \langle \frac{d\phi}{dt}(t), \psi(t)\rangle + \langle \phi(t), \frac{d\psi}{dt}(t)\rangle \tag{4}$$

Proof. First notice that (3) means that

$$\lim_{h \to 0} \left\| \frac{d\phi}{dt}(t) - \frac{\phi(t+h) - \phi(t)}{h} \right\| = 0.$$

In particular this implies that

$$\lim_{h \to 0} \|\phi(t+h) - \phi(t)\| \leqslant \lim_{h \to 0} |h| \left(\left\| \frac{d\phi}{dt}(t) - \frac{\phi(t+h) - \phi(t)}{h} \right\| + \left\| \frac{d\phi}{dt}(t) \right\| \right) = 0,$$

and therefore $\phi(t)$ is also continuous in the norm topology of $\mathcal{H}$, and similarly for $\psi(t)$.

Consider now (4); we get

$$\frac{d}{dt}\langle \phi(t), \psi(t)\rangle = \lim_{h \to 0} \frac{\langle \phi(t+h), \psi(t+h)\rangle - \langle \phi(t), \psi(t)\rangle}{h}.$$

The term inside the limit can be decomposed as follows:

$$\begin{aligned}
&\frac{1}{h}\left(\left\langle \phi\left(t+h\right), \psi\left(t+h\right)\right\rangle - \left\langle \phi\left(t\right), \psi\left(t\right)\right\rangle\right) = \\
&\qquad = \frac{1}{h}\left(\left\langle \phi\left(t+h\right)-\phi\left(t\right), \psi\left(t+h\right)\right\rangle + \left\langle \phi\left(t\right), \psi\left(t+h\right)-\psi\left(t\right)\right\rangle\right) \\
&\qquad = \left\langle \frac{\phi\left(t+h\right)-\phi\left(t\right)}{h}, \psi\left(t+h\right)-\psi\left(t\right)\right\rangle + \left\langle \frac{\phi\left(t+h\right)-\phi\left(t\right)}{h}, \psi\left(t\right)\right\rangle \\
&\qquad\quad + \left\langle \phi\left(t\right), \frac{\psi\left(t+h\right)-\psi\left(t\right)}{h}\right\rangle.
\end{aligned}$$

We now study the limit of these three terms. The first one can be bound completely, so we can apply Cauchy-Schwarz to get

$$\begin{aligned}
\lim_{h\to 0}\left|\left\langle \frac{\phi\left(t+h\right)-\phi\left(t\right)}{h}, \psi\left(t+h\right)-\psi\left(t\right)\right\rangle\right| &\leqslant \lim_{h\to 0}\left\|\frac{\phi\left(t+h\right)-\phi\left(t\right)}{h}\right\|\left\|\psi\left(t+h\right)-\psi\left(t\right)\right\| \\
&= \lim_{h\to 0}\left\|\frac{d\phi}{dt}\left(t\right)\right\|\left\|\psi\left(t+h\right)-\psi\left(t\right)\right\| = 0.
\end{aligned}$$

For the second term one can proceed as follows. Using the fact that $\phi\left(t\right)$ is differentiable and applying Cauchy-Schwarz again we get

$$\begin{aligned}
&\lim_{h\to 0}\left|\left\langle \frac{\phi\left(t+h\right)-\phi\left(t\right)}{h}, \psi\left(t\right)\right\rangle - \left\langle \frac{d\phi}{dt}\left(t\right), \psi\left(t\right)\right\rangle\right| \leqslant \\
&\qquad \leqslant \lim_{h\to 0}\left\|\frac{\phi\left(t+h\right)-\phi\left(t\right)}{h} - \frac{d\phi}{dt}\left(t\right)\right\|\left\|\psi\left(t\right)\right\| = 0.
\end{aligned}$$

Proceeding similarly for the third term we get the result.

□

Exercise 3

Let $\mathcal{H}$ be a Hilbert space. Consider A and B bounded self-adjoint operators on $\mathcal{H}$. Prove that $\frac{1}{i\hbar}\left[A, B\right]$ is self adjoint.

Proof. Recall that in the last exercise sheet we proved that $\left(AB\right)^* = B^*A^*$ and that $\left(\alpha A\right)^* = \overline{\alpha}A^*$ for any A, B bounded operators on $\mathcal{H}$ and for any $\alpha \in \mathbb{C}$. We therefore get

$$\begin{aligned}
\left(\frac{1}{i\hbar}\left[A, B\right]\right)^* &= -\frac{1}{i\hbar}\left[A, B\right]^* = -\frac{1}{i\hbar}\left(AB - BA\right)^* = -\frac{1}{i\hbar}\left(B^*A^* - A^*B^*\right) \\
&= -\frac{1}{i\hbar}\left(BA - AB\right) = -\frac{1}{i\hbar}\left[B, A\right] = \frac{1}{i\hbar}\left[A, B\right].
\end{aligned}$$

□

Exercise 4

Consider a vector space V over $\mathbb{C}$, A, B, C linear bounded operators on V and $\alpha \in \mathbb{C}$.

a Prove that $[A, B + \alpha C] = [A, B] + \alpha [A, C]$.

b Prove that $[B, A] = -[A, B]$.

c Prove that $[A, BC] = [A, B] C + B [A, C]$.

d Prove that $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$.

Proof. To prove **a** notice that

$$\begin{aligned}[A, B + \alpha C] &= A (B + \alpha C) - (B + \alpha C) A = AB - BA + \alpha AC - \alpha CA \\ &= [A, B] + \alpha [A, C].\end{aligned}$$

To prove **b** one can see that

$$[B, A] = BA - AB = -(AB - BA) = -[A, B].$$

To prove **c** we look at the right side to get

$$\begin{aligned}[A, B] C + B [A, C] &= (AB - BA) C + B (AC - CA) \\ &= ABC - BAC + BAC - BCA = [A, BC].\end{aligned}$$

To prove **d** we notice that

$$\begin{aligned}[A, [B, C]] + [B, [C, A]] + [C, [A, B]] &= \\ &= A (BC - CB) - (BC - CB) A \\ &+ B (CA - AC) - (CA - AC) B \\ &+ C (AB - BA) - (AB - BA) C = 0.\end{aligned}$$

This implies in particular

$$[A, [B, C]] = -[B, [C, A]] - [C, [A, B]] = [[A, B], C] + [B, [A, C]].$$

□